A

Synopsis

On

Study of Some Generalized Mathematical Inequalities and its Applications

submitted to

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Study of Some Generalized Mathematical Inequalities and its Application

1 Introduction to proposed research topic:

Inequality, in mathematics, a statement of an order relationship, greater than, greater than or equal to, less than, or less than or equal to between two numbers or algebraic expressions. In other words, any statement which includes the words at least, at most, more than or less than is an inequality. Mathematical inequalities are essential to the study of mathematics as well as many related fields, and their uses are extensive.

In a sense, the history of inequalities has already been written. The book Inequalities by Hardy et al.[1] organizes known inequalities and gives correct attribution to their source.

There are many classical inequalities that are not only known but also quite relevant today. For instance, the famous inequality from Jordan[2]: if $0 < |x| < \frac{\pi}{2}$ then, $\frac{2}{\pi} \leqslant \frac{\sin x}{x} < 1$.

If the exact relationship between value and function is not certain then mathematical inequalities are extremely useful. Mathematical inequalities are useful tool for providing boundness. Inequalities are used in all the time in the world around us. They can be used to model a number of real life situations. In calculus or analysis basic definitions like limit, continuity, derivatives etc. are formulated using inequalities. In mathematics, trigonometric and hyperbolic functions are crucial. Hyperbolic functions are analogue of the ordinary trigonometric functions but defined using the hyperbola rather than the circle. Hyperbolic functions were introduced in the 1760's independently by Vincenzo Riccati and Johann Heinrich Lambart. The hyperbolic functions arise in many problems of mathematics and mathematical physics specially in the solutions of many linear differential equations such

as equation defining a catenary. During the past several years, there has been a great interest in trigonometric and hyperbolic inequalities. Inequalities involving these functions are quite useful in many branches of mathematics. There are some special types of inequalities related to the trigonometric and hyperbolic functions such as, the famous Cusa-Huygen's inequality [3], Lazarvic inequality [4], Bernoulli inequality [5]etc.

2 Review of literature and development in the subject area:

During the past several years, there has been great interest in inequalities involving trigonometric and hyperbolic functions, for more information regarding it can be found in [1,6,7,8].

Sharp bounds for useful functions play a central role in many areas of mathematics and theoretical physics. Some proofs of inequalities for getting sharp bounds are based on the L'Hôpital rule of monotonicity.

L'Hôpital Monotone Rule: Let $-\infty < a < b < \infty$, and let $f, g\colon [a,b] \to \mathbb{R}$ be continuous functions that are differentiable on (a,b) , with $f(a) = g(a) = 0$ or $f(b) = g(b) = 0$. Assume that $g'(x) \neq 0$ for each $x \in (a,b)$. If $\frac{f'}{g'}$ is increasing(decreasing) on (a,b), then so is $\frac{f}{g}$.

For some other applications of the L'Hôpital Monotone Rule in special functions, one can refer [9].

The Wilker-type inequality for trigonometric functions state that

$(\frac{\sin x}{x})^2 + \frac{tanx}{x} > 2, \qquad 0 < x < \frac{\pi}{2}$.

In [10], Zhu proved the hyperbolic version of Wilker-type inequality as follows,

$(\frac{sinhx}{x})^2 + (\frac{\tanh x}{x}) > 2, \quad x > 0.$

The unnormalized *sinc* function is defined as $\sin cx = \frac{sinx}{x}; \quad x \neq 0$. It

finds applications in many branches of applied science, such as difference equations, approximation theory etc. More information regarding it can be found in [11].

One of the most remarkable inequality involving the unnormalized *sinc* function is given by,

$\frac{\sin x}{x} \leq \frac{2+\cos x}{3}; \quad 0 \leq x \leq \frac{\pi}{2}$.

This inequality is known as Cusa-Huygen's inequality [12, 13] in the literature. This inequality has been sharpened and extended in many ways by researchers [13].

From basic calculus we know that,

$$\arcsin x = \int_0^x \frac{1}{\sqrt{(1-t^2)}}\, dt, \qquad 0 \leqslant x \leqslant 1.$$

In 1995, Lindqvist [14] introduced a p-generalization of the *sine* function, which is denoted by $\sin_p(x)$ and defined as the inverse of the function

$$\arcsin_p x = \int_0^x \frac{1}{(1-t^p)^{\frac{1}{p}}}\, dt, \qquad 0 \leqslant x \leqslant 1, \qquad 1 < p < \infty$$

therefore, $\frac{\pi_p}{2} = \arcsin_p(1) = \int_0^1 \frac{1}{(1-t^p)^{\frac{1}{p}}}\, dt$

The inverse of $\arcsin_p$ on $[0, \frac{\pi_p}{2}]$ is called the generalized sine function. The function $\sin_p$ coincides with the ordinary *sine* function when $p = 2$. It is easy to see that the function $\sin_p$ is strictly increasing and concave on $[0, \frac{\pi_p}{2}]$. In the same way, we can define the generalized *cosine* function, the generalized tangent function and their inverses, and also the corresponding hyperbolic functions. The generalized *sine* function occurs as an eigenfunction of the Dirichlet problem for the one-dimensional p-Laplacian. There are several different definitions for these generalized trigonometric and hyperbolic functions [15-17]. These generalized functions are similar to the classical functions in various aspects.

The generalized Wilker-type inequality for hyperbolic functions is,

For $p > 1$ and $x > 0$,

$\left(\frac{\sinh_p(x)}{x}\right)^2 + \left(\frac{\tanh_p(x)}{x}\right) > 2$

The Huygens-type inequalities [18] for the generalized trigonometric and hyperbolic functions is,

let $p > 1$ then the following inequalities hold,

$(p+1)\frac{\sin_p(x)}{x} + \frac{1}{\cos_p(x)} > p+2, \qquad 0 < x < \frac{\pi_p}{2}$

and

$(p+1)\frac{\sinh_p(x)}{x} + \frac{1}{\cosh_p(x)} > p+2, \qquad x > 0$

3 Aims and Objectives of research:

1. To study different generalized trigonometric and hyperbolic functions and their important properties which can be useful for further developments.
2. To develop new useful results on generalized trigonometric and generalized hyperbolic functions.
3. To establish new inequalities involving generalized trigonometric and hyperbolic functions.
4. To refine old inequalities involving generalized elementary functions and trigonometric and hyperbolic functions.
5. To develop new method for obtaining existing and analytic inequalities.
6. To apply new results on generalized trigonometric and generalized hyperbolic functions and inequalities.
7. To explore inequalities involving *sinc* and hyperbolic *sinc* functions.

4 Research Methodology:

Knowledge of the concern topic will be obtained by the survey of research papers, review articles, various books and journals. We will analyze results regarding this subject given in various journals, books. We will obtain different new results regarding generalized trigonometric and hyperbolic functions and inequalities. Light will be thrown on making graphical representation of generalized trigonometric and generalized hyperbolic functions and different bounds of generalized trigonometric and hyperbolic functions. Also we would like to visit different research institutes like IIT's, TIFR, SGB Amravati University, Amravati, IISc etc. to interact with eminent mathematicians, also would like to attend national, international conferences related to our research work.

5 Importance of study/Society Applications:

We hope that, the work of this research will be useful for the engineers, scientists and mathematicians, researchers at the international, national level. Basic aim of our study is to generalize and sharpen the inequalities involving the generalized trigonometric, generalized hyperbolic, *sinc* function, generalized logarithmic, exponential functions etc. There is a large scope to investigate new inequalities involving generalized trigonometric and hyperbolic functions. The present work will be useful for budding researchers.

International Status:

Many mathematicians are working in this field. Some of them are Marko Kostić (University of Novi Sad), Branko Malesevic (University of Belgrade), Christophe Chesneau (University of Caen Normandie, France), Bojan Banjac (University of Novi Sad), Horst Alzer (Waldbrol), Man Kam Kwong (Hong Kong), Barkat A. Bhayo (IBA University, Sindh, Pakistan), Xiao-Diao Chen (Hangzhou Dianzi University), etc.

National Status:

Some of the Indian mathematicians are also working on same lines. Few of them are Dr. D. B. Pachpatte(Dr.Babasaheb Ambedkar Marathwada University, Aurangabad), Ramkrishna M. Dhaigude (Government Vidarbh Institute of Science and Humanitites, Amravati), etc.

6 Plan of work:

Sr.No.	Period	Plan of the work	Targets to be Achieved
1	First year	Study of the background required for the research topic.	Good command over the subject and clear the course work.
2	Second year	Collecting the papers and analyze them.	Try to get some construction techniques about inequalities.
3	Third year	Work on some problems to be included in the thesis.	Submit at least two research papers.
4	Fourth year	Publishing some papers and submitting the thesis.	Some more papers and thesis submission.

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